ACKNOWLEDGEMENT

This research was sponsored by the National Science Foundation under grant GK-2567.

REFERENCES

1. E. M. SPARROW and E. R. G. ECKERT, Effects of superheated vapor and noncondensable gases on laminar film condensation, *A.I.Ch.E. Jl* 7, 473–477 (1961).
2. E. M. SPARROW and S. H. LIN, Condensation heat transfer in the presence of a noncondensable gas, *J. Heat Transfer* **86**, 430–436 (1964).
3. W. J. MINKOWYCZ and E. M. SPARROW, Condensation heat transfer in the presence of noncondensables, interfacial resistance, superheating, variable properties and diffusion, *Int. J. Heat Mass Transfer* **9**, 1125–1144 (1966).
4. D. F. OTHMER, The condensation of steam, *Ind. Engng Chem.* **21**, 576–583 (1929).
5. S. J. MEISENBURG, R. M. BOARTS and W. L. BADGER, The influence of small concentration of air on the steam film coefficient of heat transfer, *Trans. Am. Inst. Chem. Engrs* **31**, 622–637 (1935) and **32**, 100–104 (1936).
6. D. L. SPENCER, K. I. CHANG and H. C. MOY, Experimental investigation of stability effects in laminar film condensation on a vertical cylinder, 4th International Heat Transfer Conference, Paris, Vol. 6, Paper Cs 2.3 (1970).
7. W. NUSSELT, Die oberflachen Kondensation des Wasserdampfes, *Z.Ver.D.Ing.* **60**, 541–569 (1916).

Int. J. Heat Mass Transfer. Vol. 14, pp. 505–509. Pergamon Press 1971. Printed in Great Britain

FREE CONVECTION HEAT TRANSFER OF A LAYER OF LIQUID HEATED FROM BELOW—THE EFFECT OF MAXIMUM DENSITY

ZU-SHUNG SUN* and CHI TIEN

Department of Chemical Engineering and Metallurgy, Syracuse University, Syracuse, New York, U.S.A.

(*Received* 27 *January* 1970 *and in revised form* 10 *July* 1970)

NOMENCLATURE

a, dimensionless wave number in linear stability analysis;

A, temperature difference ratio defined as $(T_1 - T_{max})/(T_1 - T_2)$;

B, amplitude, see equations (15) and (16);

d, depth of liquid layer;

D, operator defined as d/dz^+;

f, function associated with w and θ, see equations (15) and (16);

g, gravitational acceleration;

H, quantity associated with temperature disturbance, see equation (16);

N_{Nu}, Nusselt number defined by equation (21);

N_{Ra}, Rayleigh number defined as

$$\frac{2\gamma_1 A\,\Delta Tg\,\Delta Td^3}{\nu\kappa}\left(1 + \frac{3}{2}\frac{\gamma_2}{\gamma_1}A\Delta T\right);$$

$N_{Ra_{cr}}$, critical Rayleigh number;

P, pressure;

$P_0(z)$, average pressure over x–y plane;

δp, pressure variation, equation (8);

T, temperature;

T_{max}, temperature at which the density of the liquid is maximum;

$T_0(z)$, average temperature over x–y plane;

T_1, T_2, lower and upper surface temperature of liquid layer;

ΔT, temperature difference, $T_1 - T_2$;

u_i, velocity vector;

W, velocity component x_3 (or z) coordinate;

W, quantity associated with velocity disturbance, equation (15);

x_i, coordinates.

Greek letters

κ, thermal diffusivity;

λ_1, λ_2, constants defined as

$$-\frac{1}{A}\,\frac{1 + 3\dfrac{\gamma_2}{\gamma_1}A\Delta T}{1 + \dfrac{3}{2}\dfrac{\gamma_2}{\gamma_1}A\Delta T} \text{ and } \frac{1}{A^2}\,\frac{\dfrac{3}{2}\dfrac{\gamma_2}{\gamma_1}A\Delta T}{1 + \dfrac{3}{2}\dfrac{\gamma_2}{\gamma_1}A\Delta T};$$

*Present Address: Dept. of Chemical Engineering, University of Delaware.

α_j, unit vector (0, 0, 1);
∇^2, operator defined as $\partial/\partial x_1^2 + \partial/\partial x_1^2$;
Γ, coefficient defined by equation (23);
γ_1, γ_2, temperature coefficient for density expression, see equations (1) and (2);
θ, temperature disturbance, equation (6);
ν, kinematic viscosity;
ρ, density;
ρ_{max}, maximum density;
$\delta\rho$, density difference due to temperature.

1. INTRODUCTION

IN THIS communication, an analysis on the free convection heat transfer of a horizontal liquid layer subject to a lower and upper surface temperature of T_1 and T_2, respectively, is given. The liquid possesses a maximum density at temperature T_{max} which lies within $T_1 - T_2$. Consequently, within the liquid layer, two distinct regions exist. The lower part of the liquid layer with $T_{max} < T < T_1$ possesses a positive buoyancy force and is potentially unstable while the upper part is potentially stable. The overall free convection phenomenon represents the interaction of these two sublayers.

Previous studies of the effect of maximum density on the onset of convection included those of Veronis [14], Debler [2] and Tien [10]. The following density–temperature relationship was used:

$$\rho = \rho_{max}[1 - \gamma(T - T_{max})^2]. \tag{1}$$

A more general density–temperature relationship, i.e.

$$\rho = \rho_{max}[1 - \gamma_1(T - T_{max})^2 - \gamma_2(T - T_{max})^3] \tag{2}$$

was used by the present authors [8, 9] in their recent study of the same problem. For water, equation (1) is adequate for a temperature range 0–8°C, while equation (2) remains valid for a temperature up to 30°C.

A natural extension of the stability studies is the investigation of the free convection beyond the onset of convection. Although these two problems are closely related, the mathematical nature of these two problems is different. The onset of convection study can be made with linear stability analysis with infinitesimal disturbance while the free convection problem concerns disturbances with finite amplitude. The importance of the maximum density effect on free convection is indicated by the discrepancy between the theoretically calculated melting rate of a large body of ice heated from below [11] and the experimental observation [12]. An attempt was made to reconcile the discrepancy by Yen [13] in an empirical way. However, his results are not applicable to cases without melting.

Free convection heat transfer in a horizontal layer of liquid heated from below has been studied by several investigators [3–5, 15]. Veronis [15] and Kuo [3] used finite terms of a double series expansion for the solution of the nonlinear equation. The particular form of the double series used requires both bounding surfaces being free, which cannot be realized in an experiment. The fluid was assumed to have constant thermal expansion coefficient and tedious effort was required for the computation.

Attempts were made to use the double series expansion technique for the solution of the present problem, but were abandoned because of the convergence difficulty. Instead, a more heuristic approach based on Nakagawa's hypothesis [5] that the marginal state solution obtained from the linear stability analysis can be sued in estimating the heat transfer rate for finite amplitude convection. By an indirect comparison of the results of this analysis with experimental data, the Nusselt number expression obtained in this work was found to hold true for the Rayleigh number up to five to ten times of its critical value.

2. ANALYSIS

Consider a horizontal layer of liquid of depth d, and subject to a lower and upper surface temperature of T_1 and T_2. Furthermore, assume that this liquid possesses a maximum density at temperature T_{max}, such that $T_1 < T_{max} < T_2$ or $T_2 < T_{max} < T_1$. The pertinent equations based on Boussinesq's approximations for steady-state convection can be written as

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3}$$

$$\frac{\partial}{\partial x_i}(u_i u_j) = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0}\right) - \left(1 + \frac{\delta\rho}{\rho_0}\right)\lambda_i g + \nu\nabla^2 u_i \tag{4}$$

$$\frac{\partial}{\partial x_i}(Tu_i) = \kappa\,.\,\nabla^2 T. \tag{5}$$

The coordinate x_3 (or z) extends upwards from the lower surface of the liquid layer. λ_i is the unit vector whose components are (0, 0, 1).

It is assumed that with the commencement of convective motion, all dependent variables (velocity, temperature, pressure) can be expressed as sums of two quantities, an average value dependent upon the vertical distance, and a variable part which is a function of x, y, or

$$T = T_0(z) + \theta(x_i) \tag{6}$$

$$u_i = u_i(x_i) \tag{7}$$

$$P = P_0(z) + \delta p(x_i) \tag{8}$$

$$\langle T\rangle = T_0(z) \tag{9}$$

$$\langle u_i\rangle = 0 \tag{10}$$

$$\langle P\rangle = P_0(z). \tag{11}$$

The bracket quantity represents the average value over the horizontal plane ($x_1 - x_2$ plane). By carrying out an averaging procedure suggested by Stuart [7], the following expressions are obtained:

$$\kappa T_0(z) = \kappa T_1 + \int_0^z \langle \theta w \rangle \, dz - \left[\kappa_1(T_1 - T_2) + \int_0^d \langle \theta w \rangle \, dz\right] \frac{z}{d} \quad (12)$$

$$-\left.\frac{dT_0}{dz}\right|_0 = \frac{\Delta T}{d} + \frac{1}{\kappa d}\int_0^d \langle \theta w \rangle . dz \quad (13)$$

and

$$\frac{\Delta T}{d}\int_0^d \langle \theta w \rangle \, dz + \kappa \int_0^d \langle \theta \nabla^2 \theta \rangle \, dz = \frac{1}{\kappa}\left[\int_0^d \langle \theta w \rangle^2 \, dz - \frac{1}{d}\left(\int_0^d \langle \theta w \rangle \, dz\right)^2\right]. \quad (14)$$

If the constant shape assumptions of Stuart's are used, the velocity and temperature profiles can be written as:

$$u_3 = w = BW(z)\, f(x, y) \quad (15)$$

$$\theta = BH(z)\, f(x, y). \quad (16)$$

The boundary conditions requires that

$$W = H = 0 \quad \text{at } z = 0, \text{and } d.$$

B is the amplitude. Since it is an undetermined quantity, one can assume that:

$$\langle f^2 \rangle = 1. \quad (17)$$

f is periodic along the horizontal plane and has the following properties:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -a^2 f \quad (18)$$

$$\langle f_x^2 \rangle + \langle f_y^2 \rangle = a^2. \quad (19)$$

Substituting the assumed velocity and temperature profile of equations (15) and (16) into equation (12) and with the aid of equations (17)–(19), one has

$$\frac{\Delta T}{d}\int_0^1 HW \, dz^+ - \frac{\kappa}{d^2}\int_0^1 [(DH)^2 + a^2H^2] \, dz^+ = \frac{B^2}{\kappa}\left[\int_0^1 H^2W^2 \, dz^+ - \left(\int_0^1 HW \, dz^+\right)^2\right]. \quad (20)$$

Similarly, from equation (13) and the definition of the Nusselt number, one has

$$N_{Nu} = \frac{-(dT_0/dz)_0 . d}{(\Delta T)} = 1 + \left(\frac{1}{\kappa}\right)\left(\frac{d}{\Delta T}\right).B^2\int_0^1 WH.dz^+ \quad (21)$$

The evaluation of the Nusselt number requires the knowledge of W, H as well as the amplitude B. On the other hand, the amplitude B can be evaluated from equation (20), provided H and W are known. We shall assume that a reasonable estimate of B can be made by using the expression of H and W at the marginal state, which were obtained from the linear stability analysis. This is the basic assumption of this analysis.

By using the expressions of H and W obtained previously in connection with linear stability analysis, the amplitude function B can be evaluated from equation (20), which in turn can be used for calculating N_{Nu} [equation (21)]. The final expression is found to be:

$$N_{Nu} = 1 + \Gamma\left[1 - \frac{(N_{Ra})_{cr}}{(N_{Ra})}\right] \quad (22)$$

and

$$\Gamma = \frac{[\int_0^1 (\sum_{m=1}^{\infty} C_m u_m)(\sum_{m=1}^{\infty} C_m \sin m\pi z^+)\, dz^+]^2}{\int_0^1 (\sum_{m=1}^{\infty} C_m u_m)^2 (\sum_{m=1}^{\infty} C_m \sin m\pi z^+)^2 \, dz^+ - [\int_0^1 (\sum_{m=1}^{\infty} C_m u_m)(\sum_{m=1}^{\infty} C_m \sin m\pi z^+)\, dz^+]^2}. \quad (23)$$

Equation (23) is of the same form as that obtained by Nakagawa [5] except that the coefficient Γ is not constant but dependent upon the parameters λ_1 and λ_2 which are defined in nomenclature. C_m is the coefficient of the function H. U_m is a function of Z^+. These are discussed in [9].

The coefficient Γ was evaluated for both the rigid–rigid and rigid–free cases from the results of the fourth order polynomial approximations. The results are shown in Table 1.

3. CONFIRMATION OF CALCULATED RESULTS WITH EXPERIMENTAL OBSERVATIONS

It does not appear that there has been any measurement on the heat transport across a horizontal liquid layer with the inclusion of the maximum density effect. Consequently, a direct check of the heat-transfer result of this work is not possible. Instead, its accuracy will be tested in an indirect manner. Yen *et al.* [12] measured the melting rates of a block of ice initially at uniform temperature (equal or less than the melting point) with heating at the underside. With melting front vs. time curve obtained from the experiment across which, heat is transferred into the ice block. When the underside temperature is above 4°C, the physical problem becomes the same as that considered in this work. The melting front vs time curve obtained from the experiment was found to differ from the calculated results of an earlier work [11]. This discrepancy was attributed to the fact that Tien and Yen used the O'Toole and Silveston correlation [6], for the free convective heat transport, which does not consider the density inversion effect. It was, therefore, decided to repeat some of the calculations of [11] using the heat-transfer results obtained in this work and compare it with the experimental data of Yen *et al.* [12].

Using the expression of N_{Nu} of equation (22), the pertinent equations of the melting problem of [11] are:

$$\frac{dy^+}{dt} = \frac{3}{\phi}\left[\frac{2(1+\phi)}{y^+} - \left(\frac{k_1}{k_2}\right)(S^+)^{-1} R_{\Delta T}\left\{1 + \Gamma\left(1 - \frac{1}{S^{+3}}\right)\right\}\right]$$

$$\frac{dS^+}{dt^+} = \frac{1}{\phi}\left[\left(\frac{k_1}{k_2}\right)(S^+)^{-1} R_{\Delta T}\left\{1 + \Gamma\left(1 - \frac{1}{S^{+3}}\right)\right\} - \frac{2}{y^+}\right]. \quad (25)$$

The initial conditions are:

$$S^+ = 1$$
$$y^+ = y_c^+ = f_c^+ - 1 \text{ at } t^+ = t_c^+ \quad (26)$$

and

$$y_c^+ = \sqrt{\left(\frac{\kappa_2}{\kappa_1}\right)}\pi\left(\frac{1}{\lambda}\right)\operatorname{erfc}\left[\sqrt{\left(\frac{\kappa_1}{\kappa_2}\right)}\lambda\right]\exp\left(\frac{\kappa_1}{\kappa_2}\lambda^2\right) \quad (27)$$

where S^+ is the dimensionless melting front, defined as S/S_c. The meanings of other symbols can be found from [11].

The melting front–time curve for a number of cases have been obtained numerically and the results are shown in Fig. 1. It should be emphasized that, strictly speaking, equation (22) is valid in the limit $N_{Ra} \to N_{Ra_{cr}}$. In practice, this implies that the heat-transfer results obtained in this work are correct for small values of $[(N_{Ra}/N_{Ra_{cr}}) - 1]$. For this reason, the S^+ vs. t^+ curves were terminated when S^+ reaches 3, since the heat-transfer expression of equation (22) is unlikely to be valid beyond this point. Also included in Fig. 1 are the experimental data of [12]. In this regard, the proper definition of S_c, used in [12] should be given as

$$S_c = |(N_{Ra_{cr}})\ \nu_1\kappa_1/2\gamma_1 g(T_i - T_m)^2 A(1 + \alpha_2|^{\frac{1}{3}} \quad (28)$$

where

$$\alpha_2 = \frac{3}{2}\frac{\gamma_2}{\gamma_1}A(T_i - T_m). \quad (29)$$

Consequently, proper correction was made in transmitting the experimental data of [12], to Fig. 1.

The agreement between the calculated result and experimental data as shown in Fig. 1 is reasonably good, at the beginning period of melting. Significant differences are observed as S^+ increases. The comparison indicates that good agreement was observed for S^+ up to 1·7 for all cases and up to 2·2 for certain cases. Since the Rayleigh number is proportional to the cubic power of S [see equation (28), the

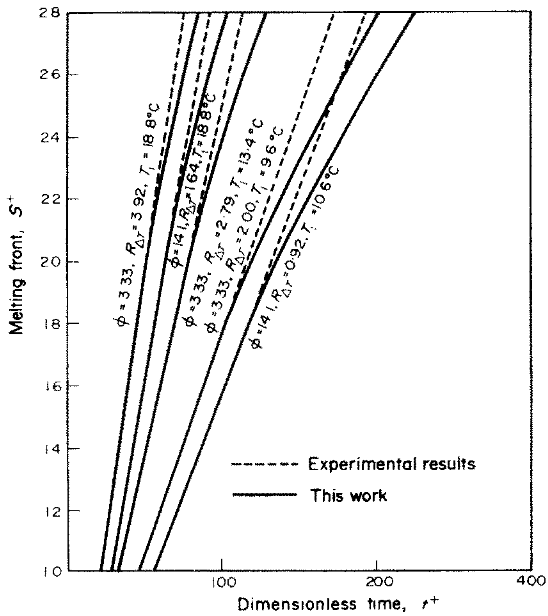

FIG. 1. Comparison of calculated melting rates with experimental work.

Nusselt number expression of equation (22) can be considered valid for $N_{Ra}/N_{Ra_{cr}}$ up to 5 ~ 10 beyond which it would not be applicable.

REFERENCES

1. S. CHANDRASEKHAR, *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press (1961).
2. W. R. DEBLER, On the analogy between thermal and rotational hydrodynamic stability, *J. Fluid Mech.* **24**, 165 (1966).
3. H. L. KUO, Solution of the non-linear equations of cellular convection and heat transport, *J. Fluid Mech.* **10**, 611 (1961).
4. W. V. R. MALKUS and GEORGE VERONIS, Finite amplitude cellular convection, *J. Fluid Mech.* **4**, 225 (1958).
5. YOSHINARI NAKAGAWA, Heat transport by convection, *Physics Fluids* **3**, 82 (1960).
6. J. L. O'TOOLE and P. L. SILVESTON, Correlations of convective heat transfer in confined horizontal layers, *Chem. Engng Prog. Symp. Ser.* **57**, No. 32, 81 (1961).
7. J. T. STUART, On the non-linear mechanics of hydrodynamic stability, *J. Fluid Mech.* **4**, 1 (1958).
8. ZU-SHUNG SUN, Thermal instability and heat transfer of a horizontal layer of liquid with maximum density and heated from below, M.S. Thesis (Chem. Eng.) Syracuse University (1968).
9. ZU-SHUNG SUN, CHI TIEN and Y. C. YEN, Thermal instability of a horizontal layer of liquid with maximum density, *A.I.Ch.E.Jl* **15**, 910 (1969).

10. Chi Tien, Thermal instability of a horizontal layer of water near 4°C, *A.I.Ch.E.Jl* **14**, 652 (1968).
11. Chi Tien and Y. C. Yen, An approximate solution of a melting problem with natural convection, *Chem. Engng Prog. Symp. Ser.* **62**, No. 64, 166 (1966).
12. Y. C. Yen, Chi Tien and Gary Sanders, An experimental study of a melting problem with natural convection, *Proc. 3rd Int. Heat Transfer Conf.* **4**, 159 (1966).
13. Y. C. Yen, Further studies on a melting problem with natural convection, *A.I.Ch.E. Jl* **82**, 4 (1967).
14. George Veronis, Penetrative convection, *Astrophysical J.* **137**, 641 (1963).
15. George Veronis, Large-amplitude Benard convection, *J. Fluid Mech.* **26**, 49 (1966).